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USE OF CHARACTERISTICS METHOD WITH CUBIC INTERPOLATION FOR UNSTEADY-FLOW COMPUTATION

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SUMMARY

The specified-time-interval (STI) scheme has been used commonly in applying the method of characteristics (MOC) to unsteady open-channel flow problems. However, with the use of STI scheme, the numerical error for the simulation results can always be induced due to the interpolation used to approximate the characteristics trajectory. Hence, in order to remedy the numerical errors caused by the interpolation, one needs to seek some kind of interpolation technique with higher-order accuracy. Instead of the linear interpolation technique, which has been used very commonly and can induce serious numerical diffusion, the Holly–Preissmann two-point method, which is a cubic interpolation technique with fourth-order of accuracy, is proposed here to integrate with the method of characteristics for the computation of one-dimensional unsteady flow in open channel. The concept of reachback and reachout in space and time directions for the characteristics is also introduced to assure the model stability. The computed results from this new model are compared with those computed by using the Preissmann four-point scheme and the multimode method of characteristics with linear interpolation.

KEY WORDS Characteristics method Cubic interpolation Unsteady flow

INTRODUCTION

For engineering purposes, the numerical model is the most efficient and economic way to solve the unsteady-flow problems. For the past two decades, a great number and variety of numerical techniques have been explored and successfully applied to simulate the unsteady flow in open channel. The Method Of Characteristics (MOC) has been well-known to have many merits on the aspects of theoretical and physical interpretation for the flow pattern. With the arrival of the modern computer, among the many other numerical schemes, this method was the first used in the numerical modelling of unsteady open-channel flows.[1,2] Later, the use of the method of characteristics for the unsteady-flow simulation has been studied extensively by many researchers, such as Vardy,[3] Wiggert and Sundquist,[4] Wylie[5] and Goldberg and Wylie.[6] Several improved versions of the method of characteristics have emerged, such as the reachback, reachout, and implicit methods. Lai[7] developed a comprehensive Multimode Method Of Characteristics (MMOC) model which combines the implicit, temporal reachback, spatial reachback, and classical schemes into one model for unsteady open-channel flow simulation. This comprehensive MMOC model can relax the Courant constraints required for the traditional explicit MOC. In addition, the accuracy of the solutions has been improved by use of the characteristics reachback concept, which allows the characteristic curves to travel beyond the present-time level.

For solving one-dimensional unsteady flow by using the method of characteristics, the two St. Venant partial differential equations must be transformed to four ordinary differential equations.

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To solve these four ordinary differential equations, there exist two general approaches. One is the characteristics–grid method, and the other is the Specified-Time-Interval (STI) method. The characteristics–grid method has the potential to give accurate solutions, but the grid system is awkward and not practical for the simulation of natural river system. The STI method has been the popular scheme for the hydraulic engineering problems. With the use of the traditional STI method, the two characteristic curves for the flow are required to fall on a rectangular grid system, and the unknowns are solved by integrating the characteristics equations along the characteristic curves. Hence, the interpolation of variables in the distance direction at the present-time level is needed. However, for most of the previous studies, the linear interpolation technique was commonly used, which leads to an inevitable smoothing of the solution. In order to reduce the numerical errors caused by the interpolation, one may need to seek an interpolation technique with higher-order of accuracy, such as quadratic or cubic interpolation technique.

In this paper the Holly–Preissmann two-point method with fourth-order of accuracy is proposed to integrate with the multimode method of characteristics (MMOC) for the unsteady-flow computation. This new model will be named cubic multimode method of characteristics (cubic MMOC). And the original MMOC model with the use of linear interpolation technique introduced by Lai[7] will be named as linear multimode method of characteristics (linear MMOC). The key to the Holly–Preissmann two-point method[8] (HP method) is based on the construction of higher-order interpolating polynomials between the dependent variables and its derivatives for two adjacent points on the spatial axis. This HP method can compute very accurately the dispersion processes in one-dimensional and two-dimensional pollutant transport problems. This method was extended to the time-line interpolation technique,[9] which allows the characteristics to intercept on the temporal axis. With the use of the reachback characteristics concept, the HP method has also been further improved and successfully applied to the dispersion problems[10,11] and surge simulation.[12]

In the following sections the mathematical and numerical formulations for the cubic MMOC model is introduced. In order to show the merits of the cubic MMOC model for the unsteady-flow simulation, a study of comparison with the Preissmann four-point model and the linear MMOC model is carried out on the basis of a hypothetical model. The properties of the cubic MMOC model are demonstrated through our examination of some key parameters, such as reachback number and Courant number.

GOVERNING EQUATIONS

The one-dimensional unsteady open-channel flow which is assumed to have uniform rectangular cross-section and to be frictionless and horizontal can be described by a differential equation set using flow velocity, u, and depth, h, as dependent variables; and distance, x, and time, t, as independent variables, as follows:

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}=g(S_0-S_f), \tag{2}$$

where S_0 denotes the bed slope, S_f is the friction slope, and g is the gravitational constant. Through some manipulation from equations (1) and (2), the following so-called characteristic equations can be obtained.[13]

$$\frac{D(u+2c)}{Dt}=g(S_0-S_f) \tag{3}$$

along

$$\left(\frac{dx}{dt}\right)_+ = u + c, \tag{4}$$

$$\frac{D(u-2c)}{Dt} = g(S_0 - S_f) \tag{5}$$

along

$$\left(\frac{dx}{dt}\right)_- = u - c, \tag{6}$$

in which $D/Dt = (\partial/\partial t) + (dx/dt)_\pm(\partial/\partial x)$ and c = celerity of gravity wave $= \sqrt{(gh)}$.

NUMERICAL ALGORITHMS

If the dependent variables u and h are assumed to be known at points r and l as shown in Figure 1, equations (3)–(6) can be solved by integrating along the characteristic curves from l to p or from r to p to obtain the four unknowns u_p, h_p, x_p, and t_p:

$$(u+2c)_p - (u+2c)_l = \int_l^p g(S_0 - S_f)\,dt, \tag{7}$$

$$x_p - x_l = \int_l^p (u+c)\,dt, \tag{8}$$

$$(u-2c)_p - (u-2c)_r = \int_r^p g(S_0 - S_f)\,dt, \tag{9}$$

$$x_p - x_r = \int_r^p (u-c)\,dt. \tag{10}$$

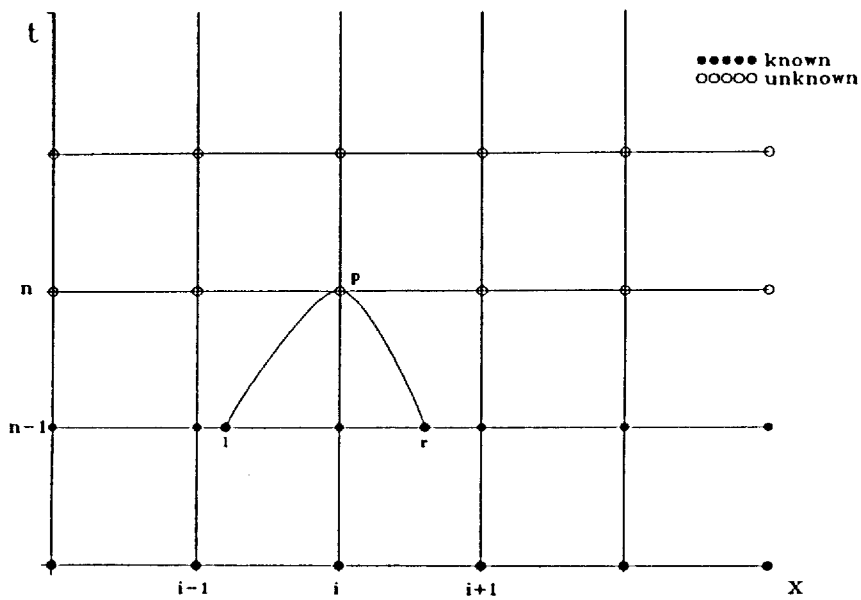


Figure 1. Grid system of classical method of characteristics

In the above equations, the integration terms appearing on the right-hand side can be approximated simply by using the trapezoidal rule:

$$\phi_{pl}=\theta\phi_p+(1-\theta)\phi_l, \tag{11}$$

$$\phi_{pr}=\theta\phi_p+(1-\theta)\phi_r, \tag{12}$$

in which ϕ can be u, c, S_0, and S_f, θ is the weighting factor, single subscript indicates a nodal point, and double subscript indicates a curve segment to which the corresponding variable or symbol belongs.

Linear interpolation scheme

To approximate ϕ_l and ϕ_r, the classical MOC model uses the linear interpolation technique:

$$\phi_l=\xi_l\phi_{i-1}^{n-1}+(1-\xi_l)\phi_i^{n-1}, \tag{13}$$

$$\phi_r=\xi_r\phi_i^{n-1}+(1-\xi_r)\phi_{i+1}^{n-1}, \tag{14}$$

in which

$$\xi_l=(u+c)_{pl}\frac{\Delta t}{\Delta x}, \tag{15}$$

$$\xi_r=(u-c)_{pr}\frac{\Delta t}{\Delta x}. \tag{16}$$

The linear interpolation technique stated above is the common method used to solve the unknown variables along the characteristics. However, it has been known that the linear approach can lead to the serious smoothing of the solution. In order to reduce the numerical smoothing caused by the linear interpolation technique, the Holly–Preissmann scheme, which, in fact, is a cubic interpolation technique, is proposed in this article.

Holly–Preissmann two-point scheme

The key to the Holly–Preissmann two-point scheme is the use of a cubic interpolation polynomial for searching the characteristics trajectory on the spatial axis. This polynomial is constructed with the use of parameters including the dependent variables and their space derivatives for two grid points on the spatial axis. Among all finite difference schemes for solving the advection portion of the dispersion equation, the Holly–Preissmann two-point method introduces the least numerical damping and phase errors.[8] In fact, the HP method used for solving dispersion equation is a kind of characteristics method in which only one characteristic curve is considered, whereas, for unsteady-flow computation, one needs to consider two characteristic curves.

To apply the Holly–Preissmann two-point scheme for evaluating the unknown variables at trajectory points l and r, one needs to introduce the spatial derivatives for flow velocity and wave celerity to construct a cubic interpolation polynomial, which can be written as follows:

$$\phi_l=a_1\phi_{i-1}^{n-1}+a_2\phi_i^{n-1}+a_3\phi_{xi-1}^{n-1}+a_4\phi_{xi}^{n-1}, \tag{17}$$

$$\phi_r=a_5\phi_i^{n-1}+a_6\phi_{i+1}^{n-1}+a_7\phi_{xi}^{n-1}+a_8\phi_{xi+1}^{n-1}, \tag{18}$$

$$\phi_{xl}=b_1\phi_{i-1}^{n-1}+b_2\phi_i^{n-1}+b_3\phi_{xi-1}^{n-1}+b_4\phi_{xi}^{n-1}, \tag{19}$$

$$\phi_{xr}=b_5\phi_i^{n-1}+b_6\phi_{i+1}^{n-1}+b_7\phi_{xi}^{n-1}+b_8\phi_{xi+1}^{n-1}, \tag{20}$$

in which ϕ represents the dependent variable, ϕ_x is the space derivative of ϕ, n is the time level, i the computational point, a_1–a_8 and b_1–b_8 are the coefficients listed in Appendix I.

Two more equations are required to evaluate u_x and c_x, which can be derived from taking the derivatives of equations (3) and (5):

$$\frac{\mathrm{D}(u_x+2c_x)}{\mathrm{D}t}=g\frac{\partial(S_0-S_f)}{\partial x}-(u_x+c_x)(u_x+2c_x), \tag{21}$$

$$\frac{\mathrm{D}(u_x-2c_x)}{\mathrm{D}t}=g\frac{\partial(S_0-S_f)}{\partial x}-(u_x-c_x)(u_x-2c_x). \tag{22}$$

By following the concept similar to that in equations (7)–(10), one can solve equations (21) and (22) to obtain the values of u_x and c_x.

Multimode scheme

In order to relax the rather severe restrictions on the size of t or x to improve the computational accuracy, several investigators have been constantly extending the MOC to a more viable and useful version. Lai[7,14] has introduced two kinds of multimode schemes. The first one combines the implicit, temporal reachback, spatial reachback and classical schemes into one. The second kind is constructed by combining the spatial reachout, temporal reachout, spatial reachback, temporal reachback and classical schemes into one.

In this study, the spatial reachback, spatial reachout, and the previously described classical MOC schemes are used to construct a multimode scheme with cubic interpolation technique. The concept of this method is shown in Figure 2. The cubic interpolation polynomials for each mode are described in detail as follows.

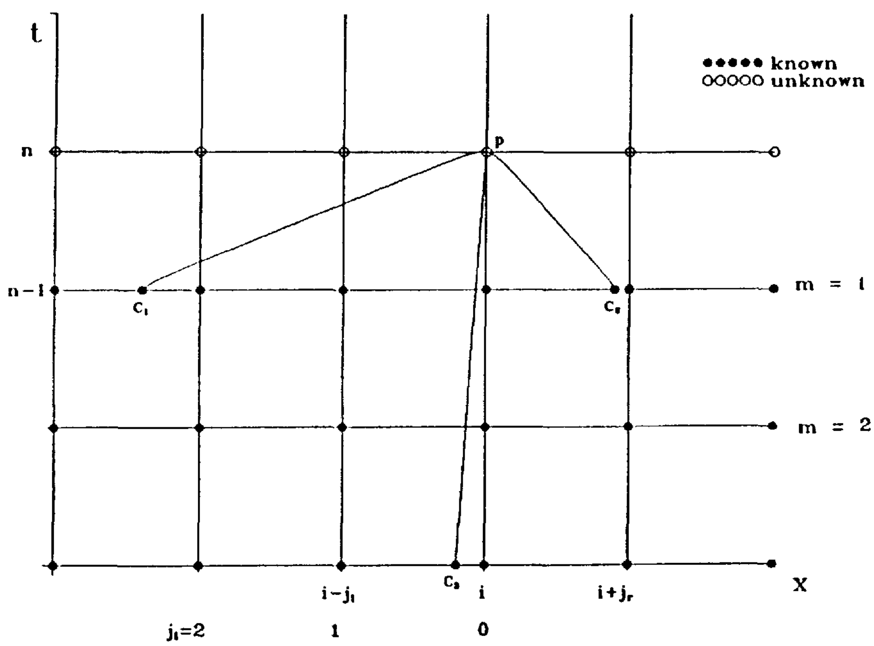


Figure 2. Grid system of multimode method of characteristics

Spatial reachback scheme. As shown in Figure 3, the characteristics are projected back beyond the present-time level and fall on the spatial axis at a certain past-time level. If the non-linearity of the system is not too strong, the spatial reachback is generally better than the classical scheme, because the intersection point is closer to the adjacent grid point.[11] The cubic interpolation polynomial for this scheme, which is slightly different from that for the classical HP method, can be written as follows:

$$\phi_l = c_1\phi_{i-1}^{n-m_l} + c_2\phi_i^{n-m_l} + c_3\phi_{xi-1}^{n-m_l} + c_4\phi_{xi}^{n-m_l}, \tag{23}$$

$$\phi_r = c_5\phi_i^{n-m_r} + c_6\phi_{i+1}^{n-m_r} + c_7\phi_{xi}^{n-m_r} + c_8\phi_{xi+1}^{n-m_r}, \tag{24}$$

$$\phi_{xl} = d_1\phi_{i-1}^{n-m_l} + d_2\phi_i^{n-m_l} + d_3\phi_{xi-1}^{n-m_l} + d_4\phi_{xi}^{n-m_l}, \tag{25}$$

$$\phi_{xr} = d_5\phi_i^{n-m_r} + d_6\phi_{i+1}^{n-m_r} + d_7\phi_{xi}^{n-m_r} + d_8\phi_{xi+1}^{n-m_r}, \tag{26}$$

in which c_1–c_8 and d_1–d_8 are coefficients listed in Appendix I and m_l and m_r are the reachback numbers for characteristic curves C_+ and C_-, respectively, which can be self-explained in Figure 3.

Spatial reachout scheme. When a large ratio of $\Delta t/\Delta x$ is used, the characteristics may extend outside of the adjacent-time lines to intersect the present-time-level spatial axis at points l and r, as shown in Figure 4. The cubic interpolation polynomial needed to approximate the unknown variables at points l and r can be given as follows:

$$\phi_l = e_1\phi_{i-j_l}^{n-1} + e_2\phi_i^{n-1} + e_3\phi_{xi-j_l}^{n-1} + e_4\phi_{xi}^{n-1}, \tag{27}$$

$$\phi_r = e_5\phi_i^{n-1} + e_6\phi_{i+j_r}^{n-1} + e_7\phi_{xi}^{n-1} + e_8\phi_{xi+j_r}^{n-1}, \tag{28}$$

$$\phi_{xl} = f_1\phi_{i-j_l}^{n-1} + f_2\phi_i^{n-1} + f_3\phi_{xi-j_l}^{n-1} + f_4\phi_{xi}^{n-1}, \tag{29}$$

$$\phi_{xr} = f_5\phi_i^{n-1} + f_6\phi_{i+j_r}^{n-1} + f_7\phi_{xi}^{n-1} + f_8\phi_{xi+j_r}^{n-1}, \tag{30}$$

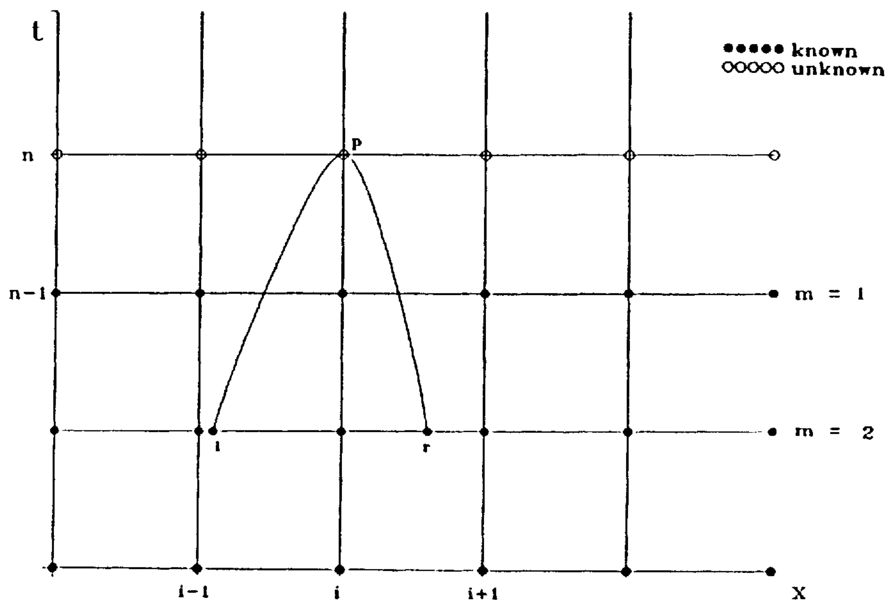


Figure 3. Grid system of space reachback method of characteristics

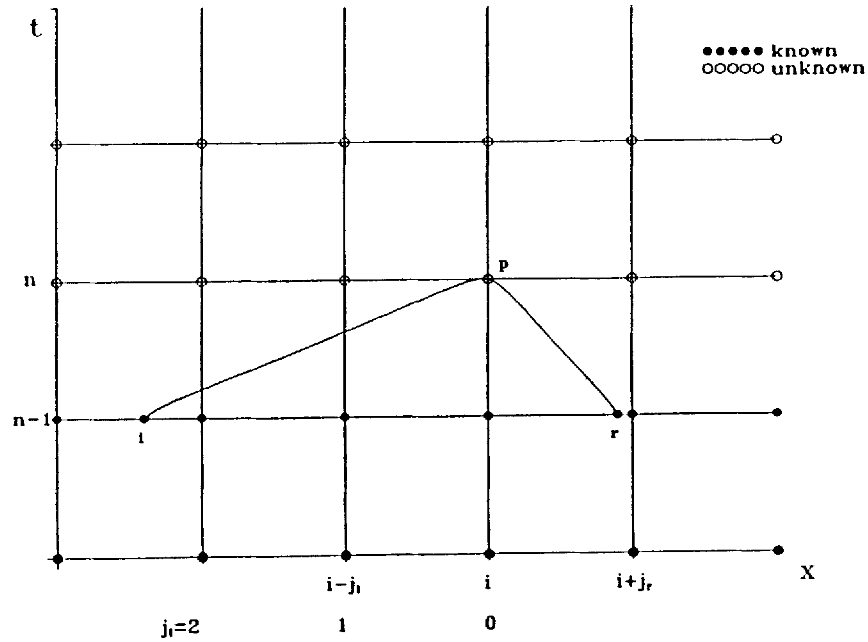


Figure 4. Grid system of space reachout method of characteristics

where e_1–e_8 and f_1–f_8 are coefficients listed in Appendix I and j_l and j_r are the reachout numbers for characteristic curves C_+ and C_-, respectively, as shown in Figure 4.

BOUNDARY CONDITIONS

The number of conditions specified on the boundary must be equal exactly to the number of characteristics originating at that boundary. For one-dimensional unsteady-flow computation, usually two boundary conditions are needed to close the system of equations (1) and (2). When the flow is subcritical, both the upstream boundary and the downstream boundary need one condition, respectively. If the flow is supercritical, two boundary conditions need to be specified at the upstream boundary. However, for the cubic MMOC model proposed here, two additional conditions are needed, since there are two more equations introduced for the calculation of derivatives of velocity and depth. The two additional conditions are the derivatives of the velocity and depth at the boundary. Based on whether the Coutant number is greater or less than 1, one needs to use different means to obtain the derivatives of velocity and depth at the boundary.

If the characteristics project outside from the adjacent-time lines to intersect on the time line of the boundaries, which is shown in Figure 5, at the upstream boundary, the cubic interpolation polynomial can be written as

$$\phi_l = g_1 \phi_0^{n-1} + g_2 \phi_0^n + g_3 \phi_{t0}^{n-1} + g_4 \phi_{t0}^n, \tag{31}$$

$$\phi_{tl} = h_1 \phi_0^{n-1} + h_2 \phi_0^n + h_3 \phi_{t0}^{n-1} + h_4 \phi_{t0}^n, \tag{32}$$

in which g_1–g_4 and h_1–h_4 are coefficients, 0 represents the upstream boundary point, ϕ_t is the time derivative of the dependent variable. At the downstream boundary, the polynomial can be given

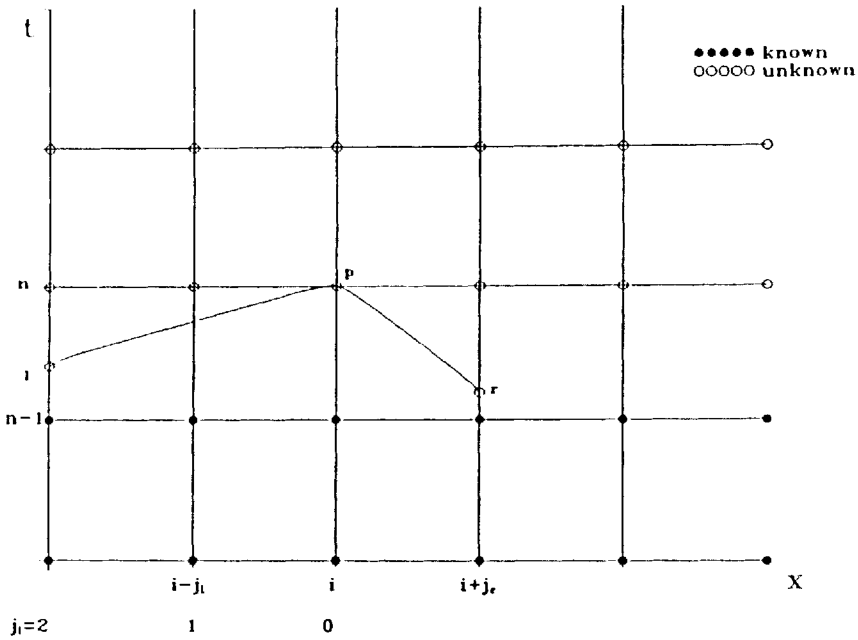


Figure 5. Grid system of characteristics reachout at the boundary

as follows:

$$\phi_r = g_5\phi_N^{n-1} + g_6\phi_N^n + g_7\phi_{tN}^{n-1} + g_8\phi_{tN}^n, \tag{33}$$

$$\phi_{tr} = h_5\phi_N^{n-1} + h_6\phi_N^n + h_7\phi_{tN}^{n-1} + h_8\phi_{tN}^n, \tag{34}$$

in which g_5–g_8 and h_5–h_8 are coefficients listed in Appendix I and N represents the downstream point.

However, the characteristic equations (21) and (22) need not the time derivative but the space derivative. Hence, the time derivatives for variables appearing in equations (31)–(34) have to be transformed into the spatial derivatives. From equations (1) and (2), the spatial derivatives for velocity and depth can be obtained as follows:

$$u_x = \frac{u(u_t) - gu(S_0 - S_f) - g(h_t)}{gh - u^2}, \tag{35}$$

$$h_x = \frac{u(h_t) - gh(S_0 - S_f) - g(u_t)}{gh - u^2}. \tag{36}$$

The time-reachout technique described above can also be used for the internal points along the channel, which, in fact, was the so-called implicit time-reachout method of characteristics.

If the Courant number is less than 1, the values of space derivatives at the boundaries can be obtained by using the forward or backward difference scheme. If the condition is specified at the upstream boundary, the forward scheme is used:

$$\phi_{x0} = \frac{\phi_1^n - \phi_0^n}{\Delta x}, \tag{37}$$

in which 0 indicates the upstream boundary point and 1 denotes the neighbouring point of upstream boundary.

If the condition is specified at the downstream boundary, the backward scheme is used:

$$\phi_{xN}=\frac{\phi_N^n-\phi_{N-1}^n}{\Delta x}, \tag{38}$$

in which N denotes the downstream boundary point.

For supercritical flow, all of these four conditions, including the specified values of velocity, depth and their derivatives, have to be assigned at the upstream boundary. For subcritical flow, the upstream boundary needs two conditions, and the downstream boundary needs the other two conditions. Here, the known velocity variation and its derivative are specified at the upstream boundary. At the downstream, the depth variation and its derivative are specified.

NUMERICAL STABILITY

The cubic MMOC model described previously combines the concept of multimode and the cubic interpolation technique. The stability analysis for the multimode method of characteristics has been carried out by Lai.[7] The stability property for the cubic interpolation technique integrating with the reachback concept used for the dispersion simulation and surge-wave computation has been discussed by Yang and Hsu[10] and Yang *et al.*,[12] respectively. Hence, no further detailed analysis will be performed here. Only the analysis results will be summarized in the following, to show the merits of the reachback concept and the cubic interpolation technique.

From Yang and Hsu's analysis[10] for the dispersion problems, it has been known that the reachback concept coupling with HP method can provide a better simulation. They concluded that the larger the value of the reachback number used, the better are the results obtained with less numerical errors. Lai[7] has found out the similar conclusion in his study of using the multimode method of characteristics for the unsteady-flow simulation. Lai used only the linear interpolation technique to couple with the multimode method of characteristics, but one has already been able to observe that the significant reachback effect on improving the accuracy of the results for the unsteady-flow problems. In addition, Lai[7] has also pointed out that the multimode model relaxes the Courant constraint needed for the classical MOC model. Later on, Yang *et al.*[12] have integrated the Preissmann's four-point scheme and the reachback Holly-Preissmann two-point scheme for the surge-wave simulation. They have also found that both, the reachback and the cubic interpolation, techniques can significantly improve the accuracy of the simulation results of the surge-wave movement.

In summarizing the stability study analysed by the previous investigators, one may be able to conclude that the reachback and the Holly-Preissmann two-point techniques have a similar effect in improving the simulation accuracy for the dispersion, unsteady-flow and surge-wave problems. No Neuman's stability analysis will be carried out here for the Holly-Preissmann two-point scheme used to study the unsteady-flow problem. But its merits will be investigated through the following comparison studies and parameter examination. From the previous study it has been known that, for the characteristics type method, an increase of reachback number can give a better solution accuracy. Hence, one may be able to verify that the cubic MMOC should provide the more accurate simulation under the following two conditions. One is that the results from the linear MMOC model with the increase of the reachback number should be approaching to those from the cubic MMOC without using the reachback technique. The other is that the cubic MMOC should not be too sensitive to the variation of reachback number.

Therefore, here the comparison will be performed to compare the results from the linear MMOC model with various reachback number to those from the cubic MMOC model without using reachback technique. It is expected that the simulation results from the linear MMOC model with the increase of reachback number will be getting close to those from the cubic MMOC model.

DEMONSTRATION AND EVALUATION

A hypothetical case is constructed to demonstrate the newly proposed cubic MMOC model for the unsteady flow computation. A uniform subcritical flow is assumed to take place in a rectangular prismatic channel. The initial discharge per unit width is $q_0 = 1{\cdot}5\ \mathrm{m^2\,s^{-1}}$. A discharge hydrograph of unit width $q = q_0 + q_w[1 - \cos(2t/T)]\ \mathrm{m^2\,s^{-1}}$, in which $q_w = 0{\cdot}25\ \mathrm{m^2\,s^{-1}}$ and T is the period of time that occurs at the upstream boundary. The downstream boundary is kept with uniform-flow condition. The Chezy's coefficient used for this study case is equal to 58.

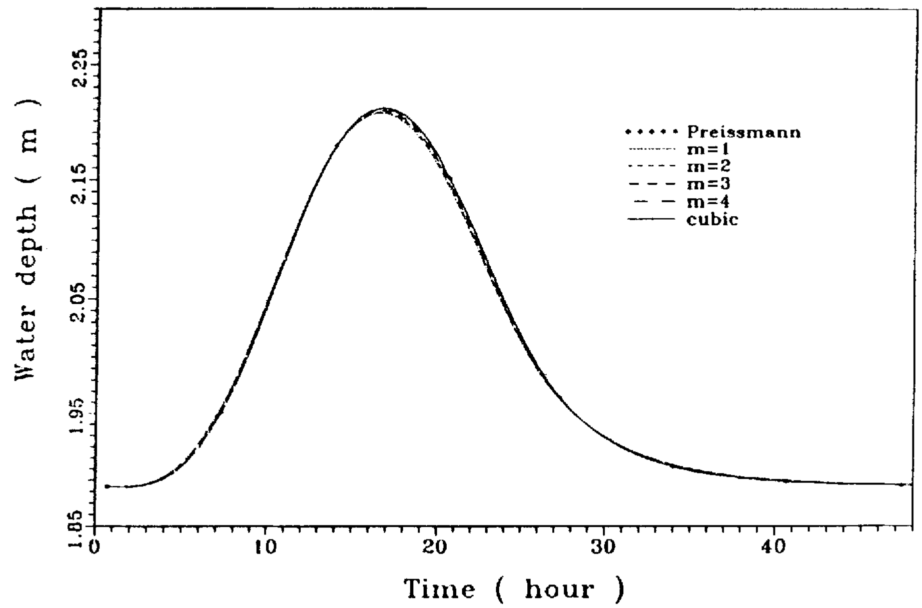
In order to show the merits of the cubic MMOC model, the comparison among the new model, the linear MMOC model, and the Preissmann four-point model is carried out. In addition, the examination on the effect of some key factors consisting of reachback number and Courant number is also performed to evaluate the applicability of the cubic MMOC model.

Comparison study

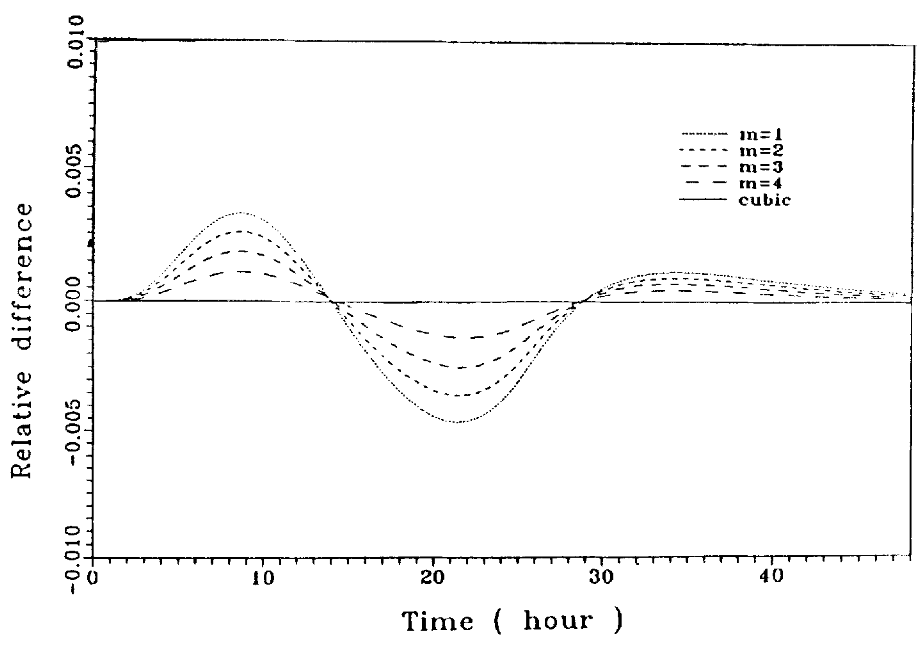
Figure 6(a) shows the change of water depth with respect to time at the section of 20 km downstream from the upstream boundary. From Figure 6(a), one can hardly tell the difference among those results computed from the various models used. Therefore, relative-difference results are computed and shown in Figure 6(b). The difference between the results computed from the Preissmann four-point scheme and that computed from the other model, which then is divided by the initial water depth, is defined as relative difference. Now one can see very clearly that the results from the cubic MMOC model with zero reachback number is consistent with that from the Preissmann four-point model. When the linear MMOC model is used, one can see that the increase of the reachback number will reduce the relative difference. This means that the results computed from the linear MMOC model with larger reachback number will approach that from the cubic MMOC and the Preissmann four-point models. It has been known that for the linear MMOC model the increase of the reachback number will increase the accuracy of the results. Hence, as the reachback number keeps increasing, it is expected that the results should finally approach those from the cubic MMOC model. Therefore, the merits of cubic MMOC model are apparently seen here.

Effect of reachback number, m

Again, from Figure 6 one knows that, for the linear MMOC model, the increase of reachback number does improve the accuracy of the simulated results significantly. It can be seen from Figure 6(b) that the relative difference obviously decreases as the reachback number increases. From the previous discussion, one has already known that the cubic MMOC model may give very accurate results even with the reachback number $m = 0$ (i.e. no reachback technique used). In order to examine the effect of the reachback number to the cubic MMOC model, a few test runs by using cubic MMOC model with various reachback numbers were performed as well. The simulated depth variation with respect to time at the section of 20 km downstream is shown in Figure 7(a). From Figure 7(a), the results computed by the cubic MMOC model with various reachback numbers and those computed by the Preissmann's four-point model are almost



(a)



(b)

Figure 6. Results computed from linear MMOC model with various m, cubic MMOC model and Preissmann four-point model ($x = 20$ km): (a) depth variation with respect to time; (b) relative difference

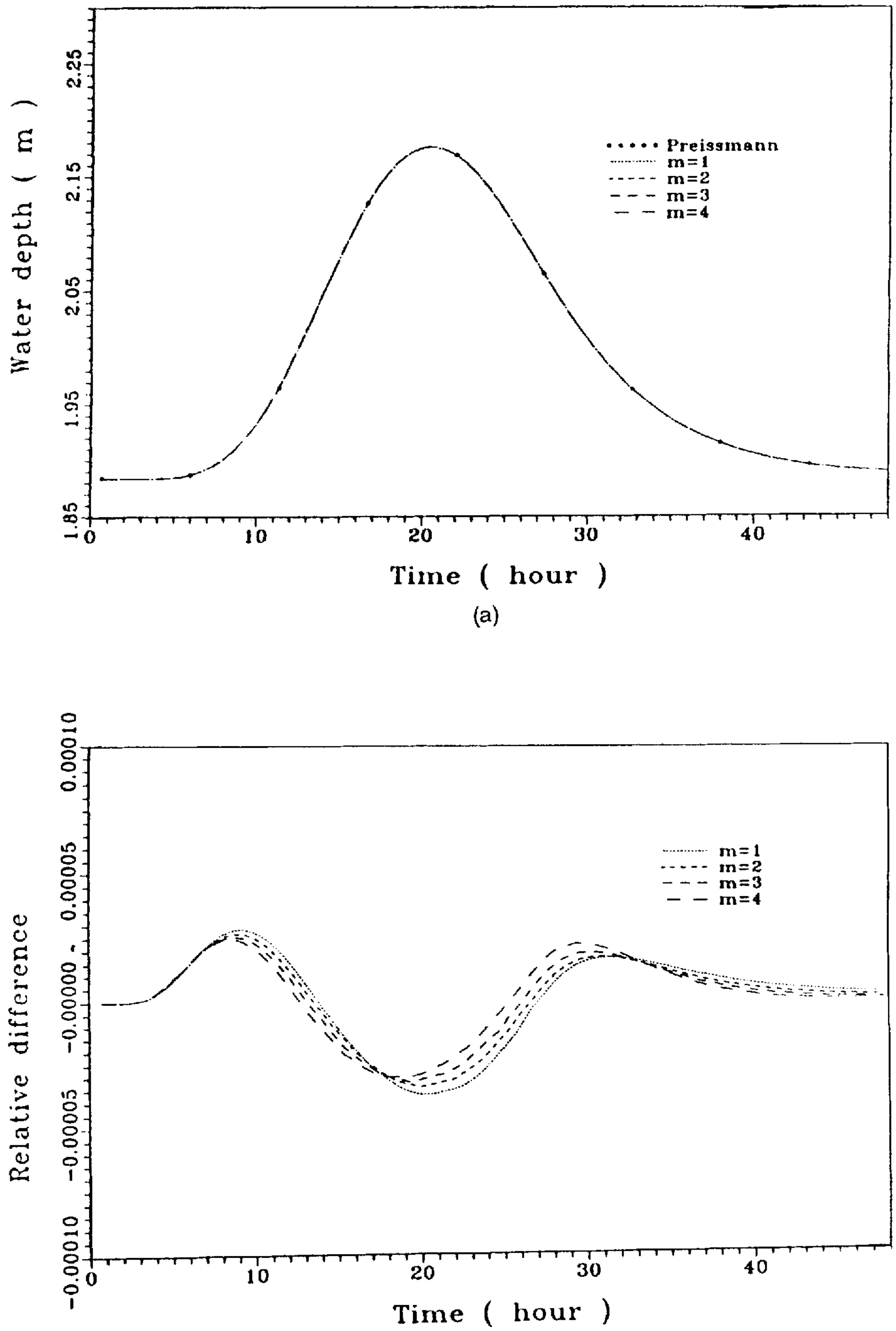
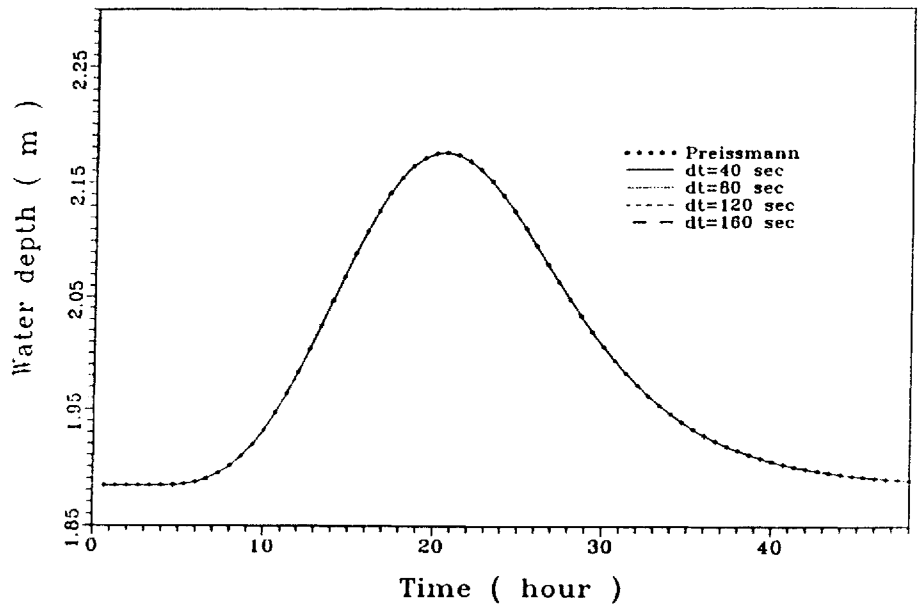
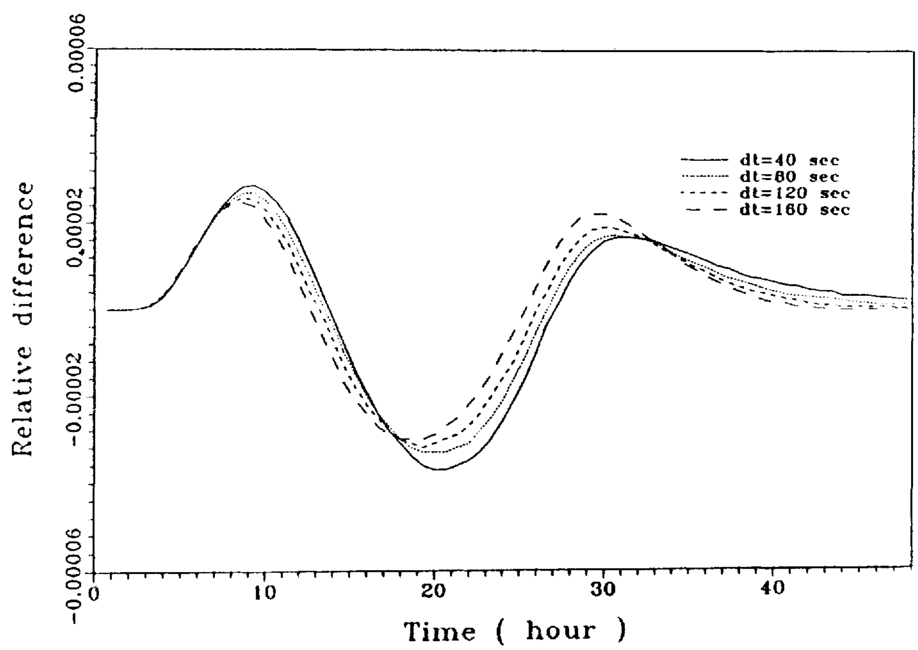


Figure 7. Results computed from cubic MMOC model with various m and Preissmann four-point model ($x = 20$ km): (a) depth variation with respect to time; (b) relative difference



(a)



(b)

Figure 8. Results computed from cubic MMOC model with various time intervals and Preissmann four-point model ($x=20$ km): (a) depth variation with respect to time; (b) relative difference

identical. The relative difference for these results, given in Figure 7(a), is shown in Figure 7(b). From Figure 7(b), one still can hardly tell such a small relative difference, of the order of less than 10^{-5}. Obviously, the reachback characteristics concept influences the results very little. The accuracy of the results has been improved very little with the increase of the reachback number. This may lead to the conclusion that the MMOC model coupled with the Holly–Preissmann two-point scheme improves significantly the accuracy of the results and the reachback concept may not be needed as for this purpose. The difficulty for establishing the initial condition existing for the reachback type model, such as linear MMOC model, can be avoided by using the cubic MMOC model. In addition, the program coding becomes simpler since now one does not need to keep tracing back the position of the reachback characteristics, and therefore, the memory of the program required will also be reduced drastically.

Effect of Courant number, Cr

In using the MOC model for the unsteady-flow computation, one is concerned most with the stability and the accuracy as influenced by the Courant number. From the previous description of the stability analysis, it has been known that the use of the MMOC model for the unsteady-flow computation is unconditionally stable no matter how the Courant number is varied.[7] Hence, no examination of the effect of the Courant number on the model stability will be performed here. Nevertheless, one may still be interested in knowing how the Courant number influences the accuracy of simulation results. Several test runs with various time intervals have been conducted to demonstrate the effect of the Courant number on the accuracy of the results. The comparison results are shown in Figures 8(a) and (b). From Figure 8(a), one can tell that the simulated depths from the use of various time intervals are almost identical. From Figure 8(b), one can observe that the relative difference remains in the range of less than 10^{-5}. So, again one may conclude that the use of cubic MMOC model for the unsteady-flow computation is not sensitive to the variation of Courant number. In other words, this means that the cubic MMOC model is quite stable and accurate. As far as the practical application is concerned, the insensitivity to the Courant number is quite a preferable condition, since one can hardly confine the range of Courant number due to the complex geometric pattern of the natural river channel.

CONCLUSIONS

A new method (cubic MMOC) which combines the multimode method of characteristics and the Holly–Preissmann two-point fourth-order scheme is introduced in this article for the computation of unsteady flow in open channel. From the comparison studies and the parameters examination carried out previously for a hypothetical model, one may be able to summarize the following conclusions:

(1) By using linear MMOC model for simulating the unsteady flow in open channel, as the reachback number increases the computed results will be approaching those computed from the cubic MMOC model, without considering the reachback technique.
(2) The cubic MMOC model used for the computation of unsteady flow in open channel is not sensitive to the variation of reachback number. This consequence implies that the reachback technique, which plays the key role in improving the results accuracy for the linear MMOC model, may not be needed as the cubic interpolation technique is used to couple with the MMOC model. Intuitively, one may conclude that the numerical errors induced by the interpolation can be reduced to a minimum by using the cubic MMOC model for computing the one-dimensional unsteady flow.

(3) Since the reachback technique may not be needed with the use of cubic MMOC model, the difficulty of establishing the initial condition existing for the reachback type method, such as the linear MMOC model, no longer exists for the newly introduced cubic MMOC model.
(4) The cubic MMOC model used for the computation of unsteady flow in open channel is not sensitive to the variation of Courant number. This means that this new model is quite stable and should be applicable to study the unsteady-flow problems in the natural complex river channel.
(5) It is evident that the cubic MMOC model is quite competitive to the Preissmann's four-point schemes.
(6) From the comparison study and parameters examination described previously, one may conclude that the cubic MMOC model can reduce the numerical errors caused by the interpolation problem and provide the more accurate results than the linear MMOC model.

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APPENDIX I

Coefficients of classical HP scheme

$$a_1=v_l^2(3-2v_l),\quad a_2=1-a_1,\quad a_3=v_l^2(1-v_l)\Delta x,\quad a_4=-v_l(1-v_l)^2\Delta x,$$
$$a_5=1-v_r^2(3-2v_r),\quad a_6=1-a_5,\quad a_7=v_r(v_r-1)^2\Delta x,\quad a_8=v_r^2(v_r-1)\Delta x,$$
$$b_1=6v_l(v_l-1)/\Delta x,\quad b_2=-b_1,\quad b_3=v_l(3v_l-2),\quad b_4=(v_l-1)(3v_l-1),$$
$$b_5=6v_r(v_r-1)/\Delta x,\quad b_6=-b_5,\quad b_7=(v_r-1)(3v_r-1),\quad b_8=v_r(3v_r-2),$$

in which v_l is the Courant number of characteristic curve C_+, $v_l=(u+c)_{pl}\Delta t/\Delta x$; and v_r is the Courant number of characteristic curve C_-, $v_r=(u-c)_{pr}\Delta t/\Delta x$.

Coefficients of spatial reachback HP scheme

The coefficients of spatial reachback HP scheme are the same as the classical ones stated above, but

$$v_l=(u+c)_{pl}m_l\frac{\Delta t}{\Delta x},\qquad v_r=(u-c)_{pr}m_r\frac{\Delta t}{\Delta x}.$$

Coefficients of spatial reachout HP scheme

Again the coefficients of spatial reachout HP scheme are as the classical ones stated above, but

$$v_l=(u+c)_{pl}\frac{\Delta t}{j_l\Delta x},\qquad v_r=(u-c)_{pr}\frac{\Delta t}{j_r\Delta x}.$$

Coefficients of temporal reachout HP scheme

$$g_1=1-\mu_l^2(3-2\mu_l),\quad g_2=1-g_1,\quad g_3=\mu_l(v_l-1)^2\Delta t,\quad g_4=\mu_l^2(\mu_l-1)\Delta t,$$

$$g_5=1-\mu_r^2(3-2\mu_r),\quad g_6=1-g_5,\quad g_7=\mu_r(\mu_r-1)^2\Delta t,\quad g_8=\mu_r^2(\mu_r-1)\Delta t,$$

$$h_1=6\mu_l(\mu_l-1)/\Delta t,\quad h_2=-h_1,\quad h_3=(\mu_l-1)(3\mu_l-1),\quad h_4=\mu_l(3\mu_l-2),$$

$$h_5=6\mu_r(\mu_r-1)/\Delta t,\quad h_6=-h_5,\quad h_7=(\mu_r-1)(3\mu_r-1),\quad h_8=\mu_r(3v_r-2),$$

where

$$\mu_l=1-\frac{j_l\Delta x}{(u+c)_{pl}\Delta t},\qquad \mu_r=1-\frac{j_r\Delta x}{(u-c)_{pr}\Delta t}.$$

APPENDIX II: NOTATIONS

a_1–a_8	coefficients
b_1–b_8	coefficients
c	celerity
c_1–c_8	coefficients
c_x	space derivative of celerity
d_1–d_8	coefficients
e_1–e_8	coefficients
f_1–f_8	coefficients
g	gravitational constant
g_1–g_8	coefficients
h	depth
h_1–h_8	coefficients
h_t	time derivative of depth
h_x	space derivative of depth
i	denotes computational points
j	reachout number
m	reachback number
n	denotes time level
u	velocity
u_t	time derivative of velocity
u_x	space derivative of velocity
S_0	bed slope
S_f	energy slope
t	time
x	distance
θ	temporal weighting factor
ϕ	denotes dependent variables
ϕ_t	time derivative of dependent variables
ϕ_x	space derivative of dependent variables
ξ	interpolation weighting factor
Δt	time interval
Δx	space interval

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